

A NEW RESTRICTION FOR INITIALLY STRESSED ELASTIC SOLIDS

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Abstract

We introduce a fundamental restriction on the strain energy function and stress tensor for initially stressed elastic solids. The restriction applies to strain energy functions W that are explicit functions of the elastic deformation gradient \mathbf{F} and initial stress $\boldsymbol{\tau}$, i.e. $W := W(\mathbf{F}, \boldsymbol{\tau})$. The restriction is a consequence of energy conservation and ensures that the predicted stress and strain energy do not depend upon an arbitrary choice of reference configuration. We call this restriction *initial stress reference independence* (ISRI). It transpires that *almost all* strain energy functions found in the literature do not satisfy ISRI, and may therefore lead to unphysical behaviour, which we illustrate via a simple example. To remedy this shortcoming we derive three strain energy functions that *do* satisfy the restriction. We also show that using *initial strain* (often from a virtual configuration) to model initial stress leads to strain energy functions that automatically satisfy ISRI. Finally, we reach the following important result: ISRI reduces the number of unknowns of the linear stress tensor of initially stressed solids. This new way of reducing the linear stress may open new pathways for the non-destructive determination of initial stresses via ultrasonic experiments, among others.

Keywords: initial stress, residual stress, constitutive equations, hyperelasticity, linear elasticity, reference independence

1 Introduction

Materials in many contexts operate under a significant level of internal stress, often called *residual stress* if the material is not subjected to any external loading. Residual stress is desirable in many circumstances; for example, living matter uses residual stress to preserve ideal mechanical conditions for its physiological function [9, 19]. In manufacturing, if residual stress is controlled, it can be used to strengthen materials such as turbine blades [20] and toughened glass [57]; however, residual stress is often problematic as it can cause materials to fail prematurely [60, 24]. *Pre-stress* is another common term, which refers to internal stress caused by an external load [35, 36, 50, 51]. In this paper, the term initial stress is used to describe any internal stress, irrespective of boundary conditions, and therefore encompasses *both* residual stress *and* pre-stress.

In both industrial and biological contexts, the origin and extent of initial stresses are often unknown. One way to determine these stresses is by measuring how they affect the elastic response of the material. In metallurgy, it is well known that residual stress can be estimated by drilling small holes into a metal and observing how they change shape [42]. Elastic waves are also used in many applications, since their behaviour is very sensitive to the initial stress in a material [13].

One alternative to link the response of the material to a very general dependence on the internal stress, therefore including initially stressed materials, is the implicit form of elasticity described by Rajagopal and coworkers [2, 38, 39], but this generality comes with the drawback of adding greater constitutive complexity. Explicit hyperelastic models are simpler and are accurate for many applications – the work of Hoger [16, 17] and Man [28, 27] has led to improved inverse methods for measuring initial stress [40, 37, 25, 1, 47, 22] and monitoring techniques [3].

The mechanical properties of a hyperelastic material can be conveniently determined from its strain energy function W , which gives the strain energy per unit volume of the initially stressed *reference* configuration. In classical elasticity, W is a function of only the elastic deformation gradient \mathbf{F} (i.e. $W := W(\mathbf{F})$). The simplest way to account for initial stresses is to allow W to depend on either the initial Cauchy stress tensor $\boldsymbol{\tau}$, or on an initial deformation gradient \mathbf{F}_0 from some stress-free configuration \mathcal{B}_0 . For the first method, $W := W(\mathbf{F}, \boldsymbol{\tau})$ [12, 45, 44], whereas for the second, $W := J_0^{-1}W_0(\mathbf{F}\mathbf{F}_0)$ [17, 21], where $J_0 = \det \mathbf{F}_0$ and W_0 is the strain energy per unit volume in \mathcal{B}_0 . In both cases, \mathbf{F} is the elastic deformation gradient from the initially stressed to the current configuration.

The two approaches each have relative advantages and disadvantages. If

measuring the initial stress is the main goal, then using $W := W(\mathbf{F}, \boldsymbol{\tau})$ is the more direct method, but requires an extra restriction – ISRI (presented below). It is also the more useful form when the initial stress is postulated or known a priori, such as assuming that the stress gradient in an arterial wall tends to be homogeneous [11]. If $W := J_0^{-1}W_0(\mathbf{F}\mathbf{F}_0)$, then the classical theory of nonlinear elasticity can be used (by taking \mathcal{B}_0 as the reference configuration), and ISRI is automatically satisfied. This form is more useful when a stress-free configuration is known, or when the exact form of the initial stress is not important. The two approaches are not equivalent because it is not always possible to deduce \mathbf{F}_0 from $\boldsymbol{\tau}$, as they are related by the equilibrium equation of the initially stressed configuration, which is a nonlinear partial differential equation in \mathbf{F}_0 . We discuss initially strained models in Section 3.

The primary purpose of this paper is to deduce a fundamental restriction on $W := W(\mathbf{F}, \boldsymbol{\tau})$, and discuss its consequences. To motivate the need for a new restriction, we show how a simple uniaxial deformation can lead to unphysical results when this restriction is ignored in Section 2.1. In Section 2.2, we derive this restriction, which follows from the fact elastic deformations conserve energy and we call it *initial stress reference independence* (ISRI), for reasons that will be clarified later. We assume the only source of anisotropy is due to the initial stress, though a more general form of ISRI could also be deduced for materials that include other sources of anisotropy. ISRI can be stated solely in terms of stress tensors, and should therefore hold for materials whose constitutive behaviour is not expressed in terms of a strain energy function.

It transpires that it is not easy to choose a strain energy function that satisfies ISRI. In fact, almost every strain energy function used in the literature to date does not satisfy it, in both finite elasticity [44, 32, 46, 31, 43, 33] and linear elasticity [27, 45]. To the authors’ knowledge, the only existing strain energy function that does satisfy ISRI is that derived in [11], which is an initially stressed incompressible neo-Hookean solid, as discussed in Section 2.3. To address this lack of valid models, we present two new strain energy functions that satisfy ISRI in Section 2.4. In Section 3, we discuss strain energy functions based on *initial strain*, and show that they automatically satisfy ISRI in Section 3.1.

Small elastic deformations on initially stressed solids lead to easier connections between the elastic response and the initial stress. This makes them ideal for establishing methods to measure initial stress. An important consequence of ISRI is that it restricts the linearised elastic stress tensor $\delta\boldsymbol{\sigma}(\mathbf{F}, \boldsymbol{\tau})$, as we discuss in Section 4. For materials subjected to *small* initial stress, we use ISRI to reduce the number of unknowns in $\delta\boldsymbol{\sigma}(\mathbf{F}, \boldsymbol{\tau})$ in Section 4.3. The

result is a reduced version of the stress tensor deduced in [27], which could ultimately improve the measurement of initial stress via ultrasonic experiments, among others.

In the literature, it is common to deduce the linear stress tensor $\delta\boldsymbol{\sigma}$ by considering an initial strain from a stress free configuration [56, 10, 23]. This approach is broadly called acousto-elasticity, and as discussed in Section 3, the resulting $\delta\boldsymbol{\sigma}$ automatically satisfies ISRI, but leads to an indirect connection between $\delta\boldsymbol{\sigma}$ and $\boldsymbol{\tau}$. In fact, acousto-elasticity was used by Tanuma and Man [55] to restrict the form of $\delta\boldsymbol{\sigma}(\mathbf{F}, \boldsymbol{\tau})$ when both strain and initial stress are small, which led them to our equation (83) (their equation (81)). In our approach we clarify that this equation must hold for every initially stressed elastic material, regardless of the origins of this stress.

2 Initial stress reference independence

The mechanical properties of an elastic material can be determined from its strain energy function W , which gives the strain energy per unit volume of the reference configuration. For an initially stressed material, W can be expressed in terms of the deformation gradient \mathbf{F} from the reference to the current configuration and $\boldsymbol{\tau}$, the Cauchy stress in the reference configuration, so that $W := W(\mathbf{F}, \boldsymbol{\tau})$. In general, W may also depend on position, but we omit this dependency for clarity. We call $\boldsymbol{\tau}$ the initial stress tensor and, when discussing constitutive choices, we will not require any specific boundary conditions in the reference configuration, in agreement with [32] (i.e the boundaries can either be loaded or unloaded).

In what follows, we assume that \mathbf{F} is within the *elastic* regime of the material, but make no assumptions about how the initial stress formed. The Cauchy stress tensor $\boldsymbol{\sigma}$ [34, 12] for an initially stressed material is given by

$$\boldsymbol{\sigma} := \boldsymbol{\sigma}(\mathbf{F}, \boldsymbol{\tau}) = J^{-1}\mathbf{F}\frac{\partial W}{\partial \mathbf{F}}(\mathbf{F}, \boldsymbol{\tau}) - p\mathbf{I}, \quad (1)$$

where $J = \det \mathbf{F}$, \mathbf{I} is the identity tensor and p is zero if the material is compressible or, otherwise, is a Lagrange multiplier associated with the incompressibility constraint $\det \mathbf{F} = 1$. We define differentiation with respect to a second-order tensor as follows:

$$\left(\frac{\partial}{\partial \mathbf{A}}\right)_{ij} = \frac{\partial}{\partial A_{ji}}. \quad (2)$$

Before moving on, we present an example where a specific choice of $W(\mathbf{F}, \boldsymbol{\tau})$ leads to two different stress responses for the same uniaxial deformation.

2.1 Motivating example

To study the influence of initial stress on the elastic response of a material, a simple strain energy function was postulated by Merodio *et al.* [32] as follows

$$W_{\text{MOR}} = \frac{\nu}{2} (\text{tr}(\mathbf{F}^T \mathbf{F}) - 3) + \frac{1}{2} (\text{tr}(\mathbf{F}^T \boldsymbol{\tau} \mathbf{F}) - \text{tr} \boldsymbol{\tau}), \quad (3)$$

where ν is a material constant, the superscript T indicates the transpose operator and tr the trace. As W_{MOR} is used for incompressible materials, the Cauchy stress (1) becomes

$$\boldsymbol{\sigma} = -p\mathbf{I} + \nu\mathbf{F}\mathbf{F}^T + \mathbf{F}\boldsymbol{\tau}\mathbf{F}^T. \quad (4)$$

Consider an initially stressed material described by Euclidean coordinates (X, Y, Z) . Suppose the initial stress takes the form of a homogeneous tension T along the X axis, and that the material is subsequently stretched along the same axis, then the components of the deformation gradient and initial stress tensor are given by

$$\mathbf{F} = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda^{-1/2} & 0 \\ 0 & 0 & \lambda^{-1/2} \end{pmatrix} \quad \text{and} \quad \boldsymbol{\tau} = \begin{pmatrix} T & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad (5)$$

where λ is the amount of stretch. Applying stress-free boundary conditions on the faces not under tension gives $p = \lambda^{-1}\nu$, which in turn leads to

$$\sigma_{11} := \sigma_{11}(\lambda, T) = \lambda^2(\nu + T) - \lambda^{-1}\nu, \quad (6)$$

which is the stress necessary to support any stretch λ given an initial tension T . We will now choose two different ways of achieving the same uniaxial stretch $\lambda = \tilde{\lambda}$ that should, *but do not*, result in the same stress when using the strain energy function (3). First, we consider a direct application of the stretch $\lambda = \tilde{\lambda}$ and assume that the initial tension is $T = \tau_0$. In this case,

$$\tilde{\sigma}_{11} = \sigma_{11}(\tilde{\lambda}, \tau_0) = \tilde{\lambda}^2(\nu + \tau_0) - \tilde{\lambda}^{-1}\nu. \quad (7)$$

We can also achieve the same stretch in two steps by taking $\tilde{\lambda} = \hat{\lambda}\bar{\lambda}$. That is, first we stretch by $\bar{\lambda}$ and then apply a further stretch $\hat{\lambda}$, as shown in Figure 1. Taking $\lambda = \bar{\lambda}$, and again using $T = \tau_0$, results in the stress

$$\bar{\sigma}_{11} = \sigma_{11}(\bar{\lambda}, \tau_0) = \bar{\lambda}^2(\nu + \tau_0) - \bar{\lambda}^{-1}\nu, \quad (8)$$

in the intermediate configuration. To further stretch the material, we take this intermediate configuration as our initially stressed reference configuration, where the initial tension is now $T = \bar{\sigma}_{11}$. Upon applying the second stretch $\hat{\lambda}$, we obtain

$$\tilde{\sigma}_{11} = \sigma_{11}(\hat{\lambda}, \bar{\sigma}_{11}) = \hat{\lambda}^2(\nu + \bar{\sigma}_{11}) - \hat{\lambda}^{-1}\nu \quad (9)$$

$$= \hat{\lambda}^2 \bar{\lambda}^2(\nu + \tau_0) + \hat{\lambda}^2 \nu - \hat{\lambda}^2 \bar{\lambda}^{-1} \nu - \hat{\lambda}^{-1} \nu. \quad (10)$$

Both (7) and (10) result from the same uniaxial deformation, so should be identical, but, upon substituting $\tilde{\lambda} = \hat{\lambda} \bar{\lambda}$ into (7), we find they are not.

If, instead of equation (4), we had used an initially strained model, e.g. an incompressible neo-Hookean $W := \mu \text{tr}(\mathbf{F}\mathbf{F}_0)/2$, then this unphysical result would not occur. However, as explained in the introduction, when the initial strain/stress are unknown, both $\boldsymbol{\tau}$ and \mathbf{F}_0 are unknown, and an explicit form $W := W(\mathbf{F}, \boldsymbol{\tau})$ leads to more direct connections between the elastic response and initial stress $\boldsymbol{\tau}$.

The unphysical behaviour illustrated by this example is typical of many of the strain energy functions of the form $W := W(\mathbf{F}, \boldsymbol{\tau})$ in the literature and highlights the need to restrict what forms of $W(\mathbf{F}, \boldsymbol{\tau})$ are physically permissible. Therefore, in the following section, we present a restriction on $W(\mathbf{F}, \boldsymbol{\tau})$ that ensures that such unphysical behaviour does not occur.

2.2 The restriction

The elastic energy stored in a material should remain constant under a rigid motion, so $W(\mathbf{F}, \boldsymbol{\tau}) = W(\mathbf{Q}\mathbf{F}, \boldsymbol{\tau})$ for every proper orthogonal tensor \mathbf{Q} (so that $\mathbf{Q}\mathbf{Q}^T = \mathbf{I}$ and $\det \mathbf{Q} = 1$). This identity can be used to show that W depends on \mathbf{F} only through the right Cauchy-Green tensor $\mathbf{C} = \mathbf{F}^T \mathbf{F}$ [34], which we use to rewrite the Cauchy stress (1) as

$$\boldsymbol{\sigma}(\mathbf{F}, \boldsymbol{\tau}) = 2J^{-1}\mathbf{F} \frac{\partial W}{\partial \mathbf{C}}(\mathbf{C}, \boldsymbol{\tau}) \mathbf{F}^T - p\mathbf{I}. \quad (11)$$

The presence of initial stress generally leads to an anisotropic material response, but for simplicity we assume that no other source of anisotropy is present. Referring to the three configurations shown in Figure 2, let the strain energy per unit volume in $\tilde{\mathcal{B}}$ be denoted by ψ . The strain energy due to the elastic deformation from \mathcal{B} to $\tilde{\mathcal{B}}$ should be the same as that due to successive elastic deformations from \mathcal{B} to $\bar{\mathcal{B}}$, then from $\bar{\mathcal{B}}$ to $\tilde{\mathcal{B}}$. In detail, taking \mathcal{B} as the reference configuration, we conclude $\psi = \tilde{J}^{-1}W(\hat{\mathbf{F}}\bar{\mathbf{F}}, \boldsymbol{\tau})$ where $\tilde{J} = \hat{J}\bar{J}$, $\hat{J} = \det \hat{\mathbf{F}}$ and $\bar{J} = \det \bar{\mathbf{F}}$, whereas if $\bar{\mathcal{B}}$ is taken as the

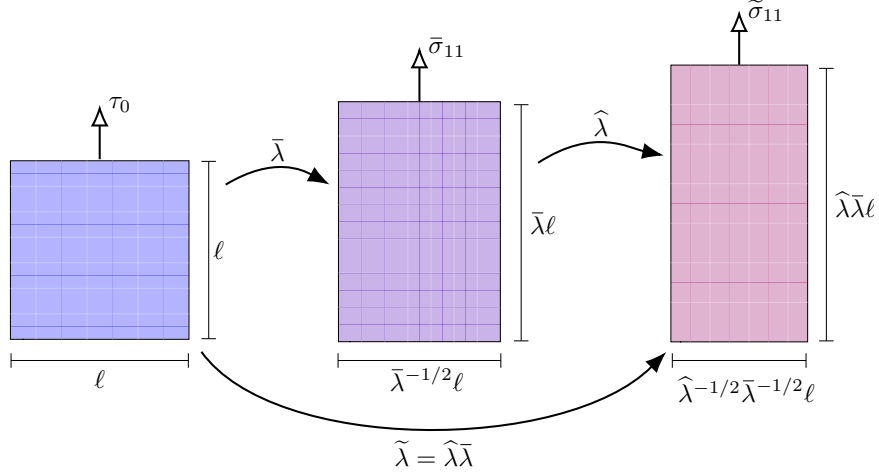


Figure 1: Uniaxial deformation of an initially stressed cube (depth not illustrated), with sides of length ℓ , into a cuboid of height $\hat{\lambda}\bar{\lambda}\ell$ and width (equal to depth) $\hat{\lambda}^{-1/2}\bar{\lambda}^{-1/2}\ell$. The hollow arrows represent the stress applied to the top boundary. The uniaxial stretch $\tilde{\lambda}$ is indicated by the bottom arrow. This stretch can also be achieved in two steps: first a stretch of $\bar{\lambda}$, then, a further stretch of $\hat{\lambda}$. The second of these stretches treats the middle configuration as its reference configuration. Both of these ways of achieving the same uniaxial stretch $\hat{\lambda}\bar{\lambda}$ should require the same stress $\tilde{\sigma}_{11}$ in the rightmost configuration.

reference configuration, we conclude $\psi = \hat{J}^{-1}W(\hat{\mathbf{F}}, \boldsymbol{\sigma}(\bar{\mathbf{F}}, \boldsymbol{\tau}))$. Since these two quantities must be equal, we therefore have

$$\boxed{W(\hat{\mathbf{F}}\bar{\mathbf{F}}, \boldsymbol{\tau}) = \bar{J}W(\hat{\mathbf{F}}, \boldsymbol{\sigma}(\bar{\mathbf{F}}, \boldsymbol{\tau})) \text{ for every } \boldsymbol{\tau}, \bar{\mathbf{F}} \text{ and } \hat{\mathbf{F}}} \quad (12)$$

where both $\bar{\mathbf{F}}$ and $\hat{\mathbf{F}}$ are associated with *elastic* deformations (which may be constrained by incompressibility). We refer to this criterion as *initial stress reference independence* (ISRI). The vast majority of initially stressed strain energy functions in the literature [45, 44, 32, 46, 31, 43, 33] do not satisfy this restriction and therefore may exhibit physically unrealistic behaviour.

When $\bar{\mathbf{F}} = \mathbf{I}$, equation (12) reduces to $W(\hat{\mathbf{F}}, \boldsymbol{\tau}) = W(\hat{\mathbf{F}}, \boldsymbol{\sigma}(\mathbf{I}, \boldsymbol{\tau}))$, which, from equation (11), is always satisfied if

$$\boldsymbol{\sigma}(\mathbf{I}, \boldsymbol{\tau}) = 2\frac{\partial W}{\partial \mathbf{C}}(\mathbf{I}, \boldsymbol{\tau}) - p\mathbf{I} = \boldsymbol{\tau}, \quad (13)$$

for every $\boldsymbol{\tau}$. We refer to this well known restriction as *initial stress compatibility*. Additionally, if $\mathbf{F} = \mathbf{Q}$, where again \mathbf{Q} is a proper orthogonal tensor representing a rigid body motion, then, using equations (11) and (13),

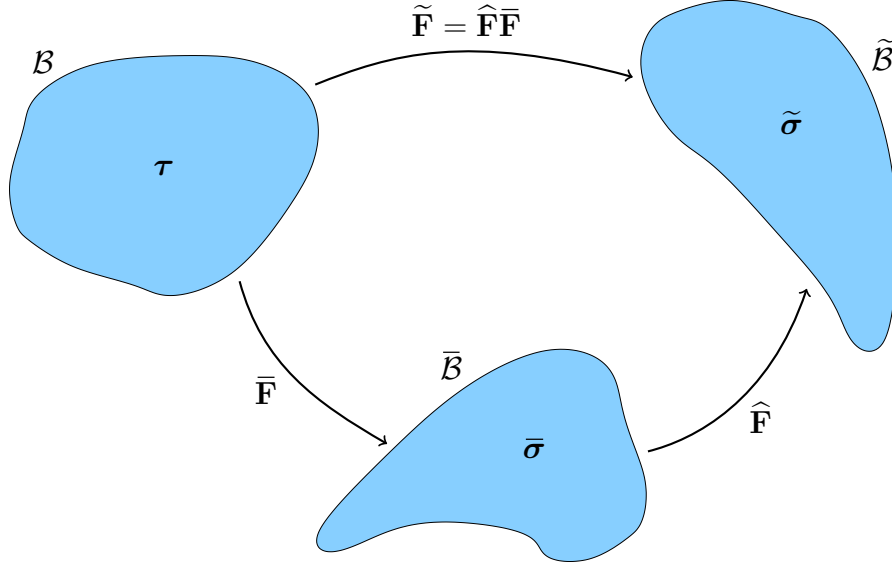


Figure 2: Deformation of initially stressed elastic solids. The stress and strain energy in $\tilde{\mathcal{B}}$ should not depend on whether \mathcal{B} or $\bar{\mathcal{B}}$ is taken as the reference configuration.

we reach $\boldsymbol{\sigma}(\mathbf{Q}, \boldsymbol{\tau}) = \mathbf{Q}\boldsymbol{\tau}\mathbf{Q}^T$. Using this result, along with $\bar{\mathbf{F}} = \mathbf{Q}$ in equation (12), we obtain

$$W(\tilde{\mathbf{F}}, \boldsymbol{\tau}) = W\left(\tilde{\mathbf{F}}\mathbf{Q}^T, \mathbf{Q}\boldsymbol{\tau}\mathbf{Q}^T\right), \quad (14)$$

where $\tilde{\mathbf{F}} = \hat{\mathbf{F}}\bar{\mathbf{F}}$. The above identity is typically used for anisotropic materials [52] and can be used to derive the following ten independent invariants [44]¹

$$I_1 = \text{tr } \mathbf{C}, \quad I_2 = \frac{1}{2}[(I_1^2 - \text{tr}(\mathbf{C}^2))], \quad I_3 = \det \mathbf{C}, \quad (15)$$

$$I_{\tau_1} = \text{tr } \boldsymbol{\tau}, \quad I_{\tau_2} = \frac{1}{2}[(I_{\tau_1}^2 - \text{tr}(\boldsymbol{\tau}^2))], \quad I_{\tau_3} = \det(\boldsymbol{\tau}), \quad (16)$$

$$J_1 = \text{tr}(\boldsymbol{\tau}\mathbf{C}), \quad J_2 = \text{tr}(\boldsymbol{\tau}\mathbf{C}^2), \quad J_3 = \text{tr}(\boldsymbol{\tau}^2\mathbf{C}), \quad J_4 = \text{tr}(\boldsymbol{\tau}^2\mathbf{C}^2). \quad (17)$$

¹Note that the invariants I_{τ_1} , I_{τ_2} and I_{τ_3} are different from, but can be expressed as linearly independent combinations of, those derived in [44].

Using these invariants, the Cauchy stress can be rewritten as

$$\begin{aligned} \boldsymbol{\sigma}(\mathbf{F}, \boldsymbol{\tau}) = & -p\mathbf{I} + \frac{1}{J} (2W_{I_1}\mathbf{B} + 2W_{I_2}(I_1\mathbf{B} - \mathbf{B}^2) + \\ & 2I_3W_{I_3}\mathbf{I} + 2W_{J_1}\mathbf{F}\boldsymbol{\tau}\mathbf{F}^T + 2W_{J_2}(\mathbf{F}\boldsymbol{\tau}\mathbf{F}^T\mathbf{B} + \mathbf{B}\mathbf{F}\boldsymbol{\tau}\mathbf{F}^T) + 2W_{J_3}\mathbf{F}\boldsymbol{\tau}^2\mathbf{F}^T + \\ & 2W_{J_4}(\mathbf{F}\boldsymbol{\tau}^2\mathbf{F}^T\mathbf{B} + \mathbf{B}\mathbf{F}\boldsymbol{\tau}^2\mathbf{F}^T)), \quad (18) \end{aligned}$$

where $\mathbf{B} = \mathbf{F}\mathbf{F}^T$ is the left Cauchy-Green tensor, $W_{I_i} = \partial W / \partial I_i$ and $W_{J_j} = \partial W / \partial J_j$, with $i \in \{1, 2, 3\}$ and $j \in \{1, 2, 3, 4\}$. For an incompressible material $I_3 = 1$ and $W_{I_3} = 0$. Note that the Cauchy stress in a standard non-linear elastic material can be obtained from (18) simply by letting W depend only on the *strain* invariants I_1 , I_2 and I_3 .

By evaluating equation (18) at $\mathbf{F} = \mathbf{I}$ we obtain another form of the initial stress compatibility equation (13):

$$\begin{aligned} \boldsymbol{\tau} = & \mathbf{I}(-\overset{\mathbf{I}}{p} + 2\overset{\mathbf{I}}{W}_{I_1} + 4\overset{\mathbf{I}}{W}_{I_2} + 2\overset{\mathbf{I}}{W}_{I_3}) \\ & + \boldsymbol{\tau}(2\overset{\mathbf{I}}{W}_{J_1} + 4\overset{\mathbf{I}}{W}_{J_2}) + \boldsymbol{\tau}^2(2\overset{\mathbf{I}}{W}_{J_3} + 4\overset{\mathbf{I}}{W}_{J_4}), \quad (19) \end{aligned}$$

where the notation $\overset{\mathbf{I}}{\cdot}$ is used to denote that \cdot is evaluated at $\mathbf{F} = \mathbf{I}$ *after* differentiation. Since this equation has to hold for *any* initial stress tensor $\boldsymbol{\tau}$, the initial stress compatibility condition is equivalent to

$$2\overset{\mathbf{I}}{W}_{I_1} + 4\overset{\mathbf{I}}{W}_{I_2} + 2\overset{\mathbf{I}}{W}_{I_3} = \overset{\mathbf{I}}{p}, \quad 2\overset{\mathbf{I}}{W}_{J_1} + 4\overset{\mathbf{I}}{W}_{J_2} = 1, \quad \overset{\mathbf{I}}{W}_{J_3} + 2\overset{\mathbf{I}}{W}_{J_4} = 0. \quad (20)$$

In the literature, W is often chosen as a simple function of the ten invariants (15,17) that satisfy initial stress compatibility (20). However, it is highly unlikely that any W chosen *a priori* will satisfy ISRI (12).

A version of ISRI can also be stated in terms of the stress tensor, without reference to a strain energy function. To do so, assume the internal stress is given by some constitutive choice $\boldsymbol{\sigma} := \boldsymbol{\sigma}(\mathbf{F}, \boldsymbol{\tau})$. Then using reasoning similar to that which led to equation (12) we find that

$$\boxed{\boldsymbol{\sigma}(\widehat{\mathbf{F}}\bar{\mathbf{F}}, \boldsymbol{\tau}) = \boldsymbol{\sigma}(\widehat{\mathbf{F}}, \boldsymbol{\sigma}(\bar{\mathbf{F}}, \boldsymbol{\tau})), \text{ for every } \boldsymbol{\tau}, \bar{\mathbf{F}} \text{ and } \widehat{\mathbf{F}}.} \quad (21)$$

This restriction states that the Cauchy stress in $\tilde{\mathcal{B}}$ should not change when a different reference configuration is selected. By choosing $\widehat{\mathbf{F}}\bar{\mathbf{F}} = \mathbf{I}$ and using equation (13), we obtain $\boldsymbol{\tau} = \boldsymbol{\sigma}(\bar{\mathbf{F}}^{-1}, \bar{\boldsymbol{\sigma}})$, where $\bar{\boldsymbol{\sigma}} = \boldsymbol{\sigma}(\bar{\mathbf{F}}, \boldsymbol{\tau})$. This restriction was derived in [11] and termed *initial stress symmetry*. It allowed a straightforward way to model the adaptive remodelling of living tissues

such as arterial walls towards an ideal target stress [5, 54]. For more details see [11] and [6].

As demonstrated in Section 2.1, strain energy functions that do not satisfy ISRI may exhibit unphysical behaviour. To the authors' knowledge, the only strain energy function in the literature to date that does satisfy ISRI is that derived in [11]. We prove this in the following section, then derive two new strain energy functions that satisfy ISRI in Section 2.4.

2.3 An incompressible strain energy function that satisfies ISRI

In a recent paper, Gower *et al.* [11] proposed the strain energy function

$$W_{\text{GCD}} = \frac{1}{2}(p_0(I_{\tau_1}, I_{\tau_2}, I_{\tau_3})I_1 + J_1 - 3\mu), \quad (22)$$

where p_0 is a function of I_{τ_1} , I_{τ_2} and I_{τ_3} given by

$$p_0 = \frac{1}{3} \left(T_3 + \frac{T_1}{T_3} - I_{\tau_1} \right), \quad (23)$$

$$T_1 = I_{\tau_1}^2 - 3I_{\tau_2}, \quad T_2 = I_{\tau_1}^3 - \frac{9}{2}I_{\tau_1}I_{\tau_2} + \frac{27}{2}(I_{\tau_3} - \mu^3), \quad (24)$$

$$T_3 = \sqrt[3]{\sqrt{T_2^2 - T_1^3} - T_2}. \quad (25)$$

One way to derive W_{GCD} is to rewrite an initially strained neo-Hookean strain energy function as an initially stressed strain energy function [11]. An alternative derivation is given in Appendix A. Using W_{GCD} in equation (18), the left side of equation (21) becomes

$$\boldsymbol{\sigma}(\hat{\mathbf{F}}\bar{\mathbf{F}}, \boldsymbol{\tau}) = p_0\hat{\mathbf{F}}\bar{\mathbf{B}}\hat{\mathbf{F}}^T - \tilde{p}\mathbf{I} + \hat{\mathbf{F}}\bar{\mathbf{F}}\boldsymbol{\tau}\bar{\mathbf{F}}^T\hat{\mathbf{F}}^T, \quad (26)$$

and the right side becomes

$$\boldsymbol{\sigma}(\hat{\mathbf{F}}, \boldsymbol{\sigma}(\bar{\mathbf{F}}, \boldsymbol{\tau})) = (p_1 - \bar{p})\hat{\mathbf{B}} + p_0\hat{\mathbf{F}}\bar{\mathbf{B}}\hat{\mathbf{F}}^T - \hat{p}\mathbf{I} + \hat{\mathbf{F}}\bar{\mathbf{F}}\boldsymbol{\tau}\bar{\mathbf{F}}^T\hat{\mathbf{F}}^T, \quad (27)$$

where p_1 is the Lagrange multiplier associated with $\bar{\mathbf{F}}$. In the Appendix A we show that $\bar{p} = p_1$, and therefore equation (27) reduces to

$$\boldsymbol{\sigma}(\hat{\mathbf{F}}, \boldsymbol{\sigma}(\bar{\mathbf{F}}, \hat{\boldsymbol{\tau}})) = p_0\hat{\mathbf{F}}\bar{\mathbf{B}}\hat{\mathbf{F}}^T - \hat{p}\mathbf{I} + \hat{\mathbf{F}}\bar{\mathbf{F}}\boldsymbol{\tau}\bar{\mathbf{F}}^T\hat{\mathbf{F}}^T. \quad (28)$$

ISRI (21) then states that

$$\boldsymbol{\sigma}(\widehat{\mathbf{F}}\bar{\mathbf{F}}, \boldsymbol{\tau}) = \boldsymbol{\sigma}(\widehat{\mathbf{F}}, \boldsymbol{\sigma}(\bar{\mathbf{F}}, \boldsymbol{\tau})) \quad \Leftrightarrow \quad \widehat{p} = \widetilde{p}. \quad (29)$$

Since equations (26) and (28) have exactly the same functional form and they must be subjected to the same boundary conditions because they both represent the Cauchy stress in $\widetilde{\mathcal{B}}$, their Lagrange multipliers must be equal (i.e. $\widehat{p} = \widetilde{p}$). Therefore, W_{GCD} *does* satisfy ISRI.

2.4 Two compressible strain energy functions that satisfy ISRI

By using the same method as that used in Appendix A to derive W_{GCD} , we have derived two new strain energy functions for compressible materials. Both are based on compressible extensions of the neo-Hookean model:

$$W_{\text{CNH1}} = \frac{\mu}{2}(I_1 - 3 - 2 \log \sqrt{I_3}) + \frac{\lambda}{2}(\log \sqrt{I_3})^2, \quad (30)$$

and

$$W_{\text{CNH2}} = \frac{\mu}{2}(I_1 - 3 - 2 \log \sqrt{I_3}) + \frac{\lambda}{2}(\sqrt{I_3} - 1)^2, \quad (31)$$

where μ and λ are the ground state first and second Lamé parameters, respectively. The initially stressed strain energy functions corresponding to these are

$$W_{\text{GSC1}} = \frac{q_1}{2}I_1 + \frac{J_1}{2} - \frac{\mu}{2K_1} \left(3 + 2 \log(K_1 \sqrt{I_3}) \right) + \frac{\lambda}{2K_1} \left(\log(K_1 \sqrt{I_3}) \right)^2 \quad (32)$$

and

$$W_{\text{GSC2}} = \frac{q_2}{2}I_1 + \frac{J_2}{2} - \frac{\mu}{2K_2} \left(3 + 2 \log(K_2 \sqrt{I_3}) \right) + \frac{\lambda}{2K_2} \left(K_2 \sqrt{I_3} - 1 \right)^2, \quad (33)$$

where q_1 , q_2 , K_1 and K_2 are functions of $I_{\boldsymbol{\tau}_1}$, $I_{\boldsymbol{\tau}_2}$ and $I_{\boldsymbol{\tau}_3}$ and can be thought of as *initial stress parameters* defined implicitly by the equations

$$\frac{\mu^3}{K_1} = q_1^3 + q_1^2 I_{\boldsymbol{\tau}_1} + q_1 I_{\boldsymbol{\tau}_2} + I_{\boldsymbol{\tau}_3}, \quad q_1 = \frac{1}{K_1}(\mu - \lambda \log K_1), \quad (34)$$

$$\frac{\mu^3}{K_2} = q_2^3 + q_2^2 I_{\boldsymbol{\tau}_1} + q_2 I_{\boldsymbol{\tau}_2} + I_{\boldsymbol{\tau}_3}, \quad q_2 = \frac{\mu}{K_2} + \lambda(1 - K_2), \quad (35)$$

where the solutions for K_1 and K_2 should both be real and such that $K_1 \rightarrow 1$ and $K_2 \rightarrow 1$ when $\boldsymbol{\tau} \rightarrow \mathbf{0}$. The Cauchy stress tensors corresponding to these strain energy functions are, respectively,

$$\boldsymbol{\sigma}_{\text{GSC1}} = \frac{1}{J} \left(q_1 \mathbf{B} + \frac{1}{K_1} (\lambda \log(JK_1) - \mu) \mathbf{I} + \mathbf{F} \boldsymbol{\tau} \mathbf{F}^T \right), \quad (36)$$

and

$$\boldsymbol{\sigma}_{\text{GSC2}} = \frac{1}{J} \left(q_2 \mathbf{B} + \left(\lambda(I_3 K_2 - J) - \frac{\mu}{K_2} \right) \mathbf{I} + \mathbf{F} \boldsymbol{\tau} \mathbf{F}^T \right). \quad (37)$$

These constitutive equations provide a simple way to study the effects of initial stress on any deformation.

3 Initially strained materials

Another way to model initial stress is via initial strain. This is normally done by including an initial deformation gradient \mathbf{F}_0 from some configuration \mathcal{B}_0 in the strain energy function $W := J_0^{-1} W_0(\mathbf{F} \mathbf{F}_0)$, where $J_0 = \det \mathbf{F}_0$ and W_0 is the strain energy per unit volume in \mathcal{B}_0 . This representation of W is a consequence of both a fundamental covariance argument [30, 26], and utilising a virtual stress-free configuration [21]. The Cauchy stress tensor is then given by [30, 26]

$$\boldsymbol{\sigma} := \boldsymbol{\sigma}(\mathbf{F} \mathbf{F}_0) = J^{-1} J_0^{-1} \mathbf{F} \frac{\partial W_0}{\partial \mathbf{F}}(\mathbf{F} \mathbf{F}_0) - p \mathbf{I}. \quad (38)$$

Usually, $W_0(\mathbf{F} \mathbf{F}_0)$ is chosen such that \mathcal{B}_0 is stress-free, that is, $\boldsymbol{\sigma}(\mathbf{I}) = \mathbf{0}$. Assuming that the initial strain is the only source of anisotropy, the strain energy can be shown to depend only on the isotropic invariants of $\mathbf{F}_0^T \mathbf{C} \mathbf{F}_0$:

$$\widehat{I}_1 = \text{tr}(\mathbf{F}_0^T \mathbf{C} \mathbf{F}_0), \quad \widehat{I}_2 = \frac{1}{2}(\widehat{I}_1^2 - \text{tr}((\mathbf{F}_0^T \mathbf{C} \mathbf{F}_0)^2)), \quad \widehat{I}_3 = \det(\mathbf{F}_0^T \mathbf{C} \mathbf{F}_0), \quad (39)$$

so that $W := J_0^{-1} W_0(\widehat{I}_1, \widehat{I}_2, \widehat{I}_3)$. These strain energy functions automatically satisfy ISRI, as shown below in Section 3.1. An example of such a strain energy function is this initially strained form of the Mooney-Rivlin strain energy function:

$$W_0 = C_1(\widehat{I}_1 \widehat{I}_3^{-1/3} - 3) + C_2(\widehat{I}_2 \widehat{I}_3^{-2/3} - 3) + C_3(\widehat{I}_3^{-1/2} - 1)^2, \quad (40)$$

where C_1 , C_2 and C_3 are material constants that must be chosen such that the body is stress free when $\mathbf{F} = \mathbf{F}_0 = \mathbf{I}$.

Taking W as a function of \mathbf{F} and $\boldsymbol{\tau}$, or of \mathbf{F} and \mathbf{F}_0 , gives two different perspectives on the same phenomenon, each being useful in different circumstances. The former is more useful when the initial *stress* is known, whereas the latter is more useful when the initial *strain* can somehow be inferred.

3.1 All initially strained materials satisfy ISRI

We have discussed, in previous sections, that it is not easy to choose a function of the form $W := W(\mathbf{F}, \boldsymbol{\tau})$ that satisfies ISRI (12). Let us consider the case of initially strained materials with

$$W = W(\mathbf{F}, \boldsymbol{\tau}) := J_0^{-1} W_0(\mathbf{F} \mathbf{F}_0), \quad \text{and} \quad \boldsymbol{\tau} = \boldsymbol{\sigma}(\mathbf{F}_0). \quad (41)$$

We will prove that if $W = W(\mathbf{F}, \boldsymbol{\tau})$ is defined as above, it satisfies ISRI for *any* choice of $W_0(\mathbf{F} \mathbf{F}_0)$. First we assume that for any W_0 and initial stress $\boldsymbol{\tau}$ there is a deformation gradient \mathbf{F}_0^2 such that

$$\boldsymbol{\tau} = \boldsymbol{\sigma}(\mathbf{F}_0) \quad \text{where} \quad \boldsymbol{\sigma}(\mathbf{F}_0) = J_0^{-1} \mathbf{F}_0 \frac{\partial W_0(\mathbf{F}_0)}{\partial \mathbf{F}_0} - p \mathbf{I}. \quad (42)$$

Next, we define an initially *stressed* strain energy function

$$W(\mathbf{F}, \boldsymbol{\tau}) = W(\mathbf{F}, \boldsymbol{\sigma}(\mathbf{F}_0)) := J_0^{-1} W_0(\mathbf{F} \mathbf{F}_0) \quad \text{for every } \mathbf{F} \text{ and } \boldsymbol{\tau}. \quad (43)$$

By substituting $\mathbf{F} = \widehat{\mathbf{F}} \bar{\mathbf{F}}$ into equation (43) we obtain

$$\begin{aligned} W(\widehat{\mathbf{F}} \bar{\mathbf{F}}, \boldsymbol{\tau}) &= J_0^{-1} W_0((\widehat{\mathbf{F}} \bar{\mathbf{F}}) \mathbf{F}_0) \\ &= \bar{J} \bar{J}^{-1} J_0^{-1} W_0(\widehat{\mathbf{F}} (\bar{\mathbf{F}} \mathbf{F}_0)) = \bar{J} W(\widehat{\mathbf{F}}, \boldsymbol{\sigma}(\bar{\mathbf{F}} \mathbf{F}_0)). \end{aligned} \quad (44)$$

Then, using equation (38), we obtain

$$\boldsymbol{\sigma}(\bar{\mathbf{F}} \mathbf{F}_0) = \bar{J}^{-1} J_0^{-1} \bar{\mathbf{F}} \frac{\partial W_0}{\partial \bar{\mathbf{F}}}(\bar{\mathbf{F}} \mathbf{F}_0) - p \mathbf{I}, \quad (45)$$

and, since $J_0^{-1} W_0(\bar{\mathbf{F}} \mathbf{F}_0) = W(\bar{\mathbf{F}}, \boldsymbol{\tau})$,

$$\boldsymbol{\sigma}(\bar{\mathbf{F}} \mathbf{F}_0) = \bar{J}^{-1} \bar{\mathbf{F}} \frac{\partial W}{\partial \bar{\mathbf{F}}}(\bar{\mathbf{F}}, \boldsymbol{\tau}) - p \mathbf{I}, \quad (46)$$

which, using equation (1), gives

$$\boldsymbol{\sigma}(\bar{\mathbf{F}} \mathbf{F}_0) = \boldsymbol{\sigma}(\bar{\mathbf{F}}, \boldsymbol{\tau}). \quad (47)$$

Substituting the above into equation (44) we obtain $W(\widehat{\mathbf{F}} \bar{\mathbf{F}}, \boldsymbol{\tau}) = \bar{J} W(\widehat{\mathbf{F}}, \boldsymbol{\sigma}(\bar{\mathbf{F}}, \boldsymbol{\tau}))$, which is the ISRI restriction (12).

Whilst such strain energy functions are guaranteed to satisfy ISRI, it is not often possible to state their dependence on the stress invariants $I_{\boldsymbol{\tau}_1}$, $I_{\boldsymbol{\tau}_2}$ and $I_{\boldsymbol{\tau}_3}$ *explicitly* (a notable exception being the strain energy function discussed in Section 2.3). Instead, it may be necessary to define that dependence *implicitly*, as is the case for the two models presented in Section 2.4.

²Note that for there to be a unique \mathbf{F}_0 , for every $\boldsymbol{\tau}$, some restrictions need to be made about the reference configuration of \mathbf{F}_0 , see [21], for example.

4 Linear elasticity with initial stress

Elastic waves in solids are highly sensitive to initial stress, and linear elastic models fit measurements from currently employed experimental techniques well. Our aim here is, in the long run, to improve these measurements by using a linearised version of ISRI (12).

In Section 4.1 we deduce the linearised stress without considering ISRI. Then, in Section 4.2, we calculate a linearised form of ISRI and discuss how to use it to restrict the linearised stress. Hoger [16, 18], Man and coworkers [28, 27] derived the equations for small initial stress, up to first order in $\boldsymbol{\tau}$. In [27] the authors remark that many experiments indicate that for small deformations the elastic stress depends linearly on the initial stress, at least for metals. Motivated by these observations, we linearise the elastic stress in both the elastic strain and initial stress in Section 4.3 and reach a reduced form for the stress (84) which adds a restriction to all previous models, to the authors' knowledge. The restriction (83) has been used before in the literature (see equation (81) from [55]) but was deduced from the context of acousto-elasticity.

4.1 Linear elastic stress

For a small elastic deformation, we can write the associated deformation gradient as $\mathbf{F} = \mathbf{I} + \nabla \mathbf{u}$, where \mathbf{u} is a small displacement. By Taylor series expanding the Cauchy stress (1) about $\mathbf{F} = \mathbf{I}$, the linearised Cauchy stress becomes

$$\delta \boldsymbol{\sigma}(\mathbf{F}, \boldsymbol{\tau}) = \boldsymbol{\tau} + \frac{\partial \boldsymbol{\sigma}}{\partial \mathbf{F}} \bigg|_{\mathbf{F}=\mathbf{I}} : \nabla \mathbf{u} + \mathcal{O}((\nabla \mathbf{u})^2), \quad (48)$$

where we have exploited the fact that $\boldsymbol{\sigma}(\mathbf{I}, \boldsymbol{\tau}) = \boldsymbol{\tau}$ and we remind the reader that $\frac{\partial}{\partial \mathbf{F}} \big|_{\mathbf{F}=\mathbf{I}}$ denotes that \cdot is evaluated at $\mathbf{F} = \mathbf{I}$ *after* differentiation. We define

$$\left(\frac{\partial \mathbf{A}}{\partial \mathbf{B}} \right)_{ijkl} = \frac{\partial A_{ij}}{\partial B_{lk}} \quad \text{and} \quad (\mathcal{C} : \mathbf{A})_{ij} = \mathcal{C}_{ij\alpha\beta} A_{\beta\alpha}, \quad (49)$$

for any second-order tensors \mathbf{A} and \mathbf{B} and fourth-order tensor \mathcal{C} , using Einstein summation convention for the repeated dummy indices α and β . Using

equations (11) and (49) it can be shown that

$$\begin{aligned} \frac{\partial \boldsymbol{\sigma}}{\partial \mathbf{F}} : \mathbf{A} &= \frac{\partial}{\partial \mathbf{F}} \left(2J^{-1} \mathbf{F} \frac{\partial W}{\partial \mathbf{C}} \mathbf{F}^T \right) \Big|_{\mathbf{F}=\mathbf{I}} : \mathbf{A} \\ &= \mathbf{A} \boldsymbol{\tau} + \boldsymbol{\tau} \mathbf{A}^T - \boldsymbol{\tau} \operatorname{tr} \mathbf{A} + 4 \frac{\partial^2 W}{\partial \mathbf{C}^2} : \mathbf{A}, \end{aligned} \quad (50)$$

for every second-order tensor \mathbf{A} , where we have exploited the fact that $2\partial W/\partial \mathbf{C}|_{\mathbf{F}=\mathbf{I}} = \boldsymbol{\tau}$ from equation (13). We now introduce the linear strain and rotation tensors:

$$\boldsymbol{\varepsilon} = \frac{1}{2}(\nabla \mathbf{u} + (\nabla \mathbf{u})^T) \quad \text{and} \quad \boldsymbol{\omega} = \frac{1}{2}(\nabla \mathbf{u} - (\nabla \mathbf{u})^T), \quad (51)$$

respectively, which satisfy $\nabla \mathbf{u} = \boldsymbol{\varepsilon} + \boldsymbol{\omega}$. Substituting $\boldsymbol{\omega}$ for \mathbf{A} in equation (50), we obtain

$$\frac{\partial \boldsymbol{\sigma}}{\partial \mathbf{F}} : \boldsymbol{\omega} = \boldsymbol{\omega} \boldsymbol{\tau} - \boldsymbol{\tau} \boldsymbol{\omega}, \quad (52)$$

since $\operatorname{tr} \boldsymbol{\omega} = 0$ and

$$\left(\frac{\partial^2 W}{\partial^2 \mathbf{C}} : \boldsymbol{\omega} \right)_{ij} = \frac{\partial^2 W}{\partial C_{ji} \partial C_{\alpha\beta}} \omega_{\alpha\beta} = -\frac{\partial^2 W}{\partial C_{ji} \partial C_{\beta\alpha}} \omega_{\beta\alpha} \Rightarrow \frac{\partial^2 W}{\partial^2 \mathbf{C}} : \boldsymbol{\omega} = \mathbf{0}, \quad (53)$$

where we have used the fact that $\boldsymbol{\omega}^T = -\boldsymbol{\omega}$ and $\mathbf{C}^T = \mathbf{C}$. Using equations (51) and (52) we can now rewrite equation (48) as

$$\delta \boldsymbol{\sigma} = \boldsymbol{\tau} + \boldsymbol{\omega} \boldsymbol{\tau} - \boldsymbol{\tau} \boldsymbol{\omega} + \frac{\partial \boldsymbol{\sigma}}{\partial \mathbf{F}} : \boldsymbol{\varepsilon} + \mathcal{O}((\nabla \mathbf{u})^2). \quad (54)$$

At this point, we do not yet know the form of $\partial \boldsymbol{\sigma}/\partial \mathbf{F}|_{\mathbf{F}=\mathbf{I}} : \boldsymbol{\varepsilon}$ explicitly. It could be calculated directly from equation (18); however, an alternative approach is to write it as a general rank two symmetric tensor in terms of $\boldsymbol{\tau}$ that is expanded up to first order in $\boldsymbol{\varepsilon}$:

$$\begin{aligned} \frac{\partial \boldsymbol{\sigma}}{\partial \mathbf{F}} : \boldsymbol{\varepsilon} &= \alpha_1 \boldsymbol{\varepsilon} + (\alpha_2 \mathbf{I} + \alpha_3 \boldsymbol{\tau} + \alpha_4 \boldsymbol{\tau}^2) \operatorname{tr}(\boldsymbol{\varepsilon}) + (\alpha_5 \mathbf{I} + \alpha_6 \boldsymbol{\tau} + \alpha_7 \boldsymbol{\tau}^2) \operatorname{tr}(\boldsymbol{\varepsilon} \boldsymbol{\tau}) \\ &\quad + \alpha_8 (\boldsymbol{\varepsilon} \boldsymbol{\tau} + \boldsymbol{\tau} \boldsymbol{\varepsilon}) + \alpha_9 (\boldsymbol{\varepsilon} \boldsymbol{\tau}^2 + \boldsymbol{\tau}^2 \boldsymbol{\varepsilon}) + \mathcal{O}((\nabla \mathbf{u})^2), \end{aligned} \quad (55)$$

where α_i , ($i = 1, \dots, 9$) are, in general, functions of I_{τ_1} , I_{τ_2} and I_{τ_3} . Note that neither $\operatorname{tr}(\boldsymbol{\varepsilon} \boldsymbol{\tau}^2)$, $\boldsymbol{\tau} \boldsymbol{\varepsilon} \boldsymbol{\tau}$, $\boldsymbol{\tau}^2 \boldsymbol{\varepsilon} \boldsymbol{\tau} + \boldsymbol{\tau} \boldsymbol{\varepsilon} \boldsymbol{\tau}^2$, nor any power of $\boldsymbol{\tau}$ higher than

two is present because they can be written as combinations of the terms already included (see Appendix B). For more details on linearising elasticity see [18, 7, 8, 44].

We now seek to restrict the parameters $\alpha_1, \dots, \alpha_9$. We begin by rearranging equation (50) and contracting it twice on the left with an arbitrary second-order tensor \mathbf{B} , to obtain

$$4\mathbf{B} : \frac{\partial^2 W}{\partial \mathbf{C}^2} : \mathbf{A} = (\mathbf{B} : \boldsymbol{\tau}) \text{tr} \mathbf{A} - \mathbf{B} : (\mathbf{A} \boldsymbol{\tau}) - \mathbf{B} : (\boldsymbol{\tau} \mathbf{A}^T) + \mathbf{B} : \frac{\partial \boldsymbol{\sigma}}{\partial \mathbf{F}} : \mathbf{A}. \quad (56)$$

Since equation (56) must hold for any \mathbf{A} and \mathbf{B} , we can swap them to obtain

$$4\mathbf{A} : \frac{\partial^2 W}{\partial \mathbf{C}^2} : \mathbf{B} = \mathbf{A} : \boldsymbol{\tau} \text{tr} \mathbf{B} - \mathbf{A} : (\mathbf{B} \boldsymbol{\tau}) - \mathbf{A} : (\boldsymbol{\tau} \mathbf{B}^T) + \mathbf{A} : \frac{\partial \boldsymbol{\sigma}}{\partial \mathbf{F}} : \mathbf{B}. \quad (57)$$

Now, due to the fact that

$$\left(\frac{\partial^2 W}{\partial^2 \mathbf{C}} \right)_{ijkl} = \left(\frac{\partial^2 W}{\partial^2 \mathbf{C}} \right)_{klij} \quad (58)$$

we must have

$$\mathbf{A} : \frac{\partial^2 W}{\partial \mathbf{C}^2} : \mathbf{B} = \mathbf{B} : \frac{\partial^2 W}{\partial \mathbf{C}^2} : \mathbf{A}, \quad (59)$$

for every \mathbf{A} and \mathbf{B} . Upon substituting equations (56) and (57) into equation (59), and assuming that \mathbf{A} and \mathbf{B} are small and symmetric, so that equation (55) holds with \mathbf{A} and \mathbf{B} substituted for $\boldsymbol{\varepsilon}$, we find that equation (59) can hold if and only if

$$\alpha_4 = \alpha_7 = 0 \quad \text{and} \quad \alpha_5 = \alpha_3 + 1. \quad (60)$$

Substituting the above into equation (54), we obtain a reduced expression for the stress:

$$\boxed{\begin{aligned} \delta \boldsymbol{\sigma} = & \boldsymbol{\tau} + \boldsymbol{\omega} \boldsymbol{\tau} - \boldsymbol{\tau} \boldsymbol{\omega} + \mathbf{I} \text{tr}(\boldsymbol{\varepsilon} \boldsymbol{\tau}) + \alpha_1 \boldsymbol{\varepsilon} + \alpha_2 \mathbf{I} \text{tr}(\boldsymbol{\varepsilon}) + \alpha_3 (\boldsymbol{\tau} \text{tr}(\boldsymbol{\varepsilon}) + \mathbf{I} \text{tr}(\boldsymbol{\varepsilon} \boldsymbol{\tau})) \\ & + \alpha_6 \boldsymbol{\tau} \text{tr}(\boldsymbol{\varepsilon} \boldsymbol{\tau}) + \alpha_8 (\boldsymbol{\varepsilon} \boldsymbol{\tau} + \boldsymbol{\tau} \boldsymbol{\varepsilon}) + \alpha_9 (\boldsymbol{\varepsilon} \boldsymbol{\tau}^2 + \boldsymbol{\tau}^2 \boldsymbol{\varepsilon}). \end{aligned}} \quad (61)$$

In Section 4.2, we discuss the linearised version of ISRI and its relationship to the linear stress tensor given in equation (61). When the initial stress is small, we are able to derive a closed-form of the linear stress that satisfies ISRI, as shown in Section 4.3.

4.1.1 Initially stressed neo-Hookean models

As an aside, we note that if the stress tensors for the initially stressed neo-Hookean models given in equations (36) and (37) are expanded for small deformations, the resulting linear stress tensors have the above form with

$$\alpha_1 = \frac{2}{K_1}(\mu - \lambda \log K_1), \quad \alpha_2 = \frac{\lambda}{K_1}, \quad \alpha_3 = -\alpha_8 = -1, \quad \alpha_6 = \alpha_9 = 0, \quad (62)$$

for the first model, and

$$\frac{\alpha_1}{2} = \frac{\mu}{K_2} + \lambda(1 - K_2), \quad \alpha_2 = \lambda(2K_2 - 1), \quad \alpha_3 = -\alpha_8 = -1, \quad \alpha_6 = \alpha_9 = 0, \quad (63)$$

for the second.

4.2 The linearised equations of ISRI

We now wish to consider the restrictions that are imposed by ISRI in the case of small deformations. We begin by differentiating equation (12) with respect to $\bar{\mathbf{F}}$ to obtain

$$\frac{\partial W}{\partial \mathbf{F}}(\hat{\mathbf{F}}\bar{\mathbf{F}}, \boldsymbol{\tau})\hat{\mathbf{F}} = \frac{\partial \bar{J}}{\partial \bar{\mathbf{F}}}W(\hat{\mathbf{F}}, \boldsymbol{\sigma}(\bar{\mathbf{F}}, \boldsymbol{\tau})) + \bar{J}\frac{\partial W}{\partial \boldsymbol{\sigma}}(\hat{\mathbf{F}}, \boldsymbol{\sigma}(\bar{\mathbf{F}}, \boldsymbol{\tau}))\frac{\partial \boldsymbol{\sigma}}{\partial \bar{\mathbf{F}}}(\bar{\mathbf{F}}, \boldsymbol{\tau}), \quad (64)$$

where $\partial/\partial \mathbf{F}$ denotes partial differentiation with respect to the first argument of the function and $\partial/\partial \boldsymbol{\sigma}$ denotes partial differentiation with respect to the second. Evaluating equation (64) at $\hat{\mathbf{F}} = \bar{\mathbf{F}} = \mathbf{I}$ and contracting twice on the right with the linear strain tensor $\boldsymbol{\varepsilon}$ gives

$$\boxed{\boldsymbol{\tau} : \boldsymbol{\varepsilon} = \text{tr } \boldsymbol{\varepsilon} \frac{\mathbf{I}}{W} + \frac{\partial W}{\partial \boldsymbol{\tau}} : \frac{\partial \boldsymbol{\sigma}}{\partial \bar{\mathbf{F}}} : \boldsymbol{\varepsilon} \quad \text{for every } \boldsymbol{\tau} \text{ and } \boldsymbol{\varepsilon},} \quad (65)$$

which was simplified using equation (13). One of the terms on the right side can be expanded using the chain rule as follows

$$\frac{\partial W}{\partial \boldsymbol{\tau}} = \beta_1 \mathbf{I} + \beta_2 \boldsymbol{\tau} + \beta_3 \boldsymbol{\tau}^2, \quad (66)$$

where

$$\beta_1 = \frac{\partial W}{\partial \text{tr } \boldsymbol{\tau}} = \frac{\partial W}{\partial I_{\tau_1}} + I_{\tau_1} \frac{\partial W}{\partial I_{\tau_2}} + I_{\tau_2} \frac{\partial W}{\partial I_{\tau_3}} + \frac{\partial W}{\partial J_1} + \frac{\partial W}{\partial J_2}, \quad (67)$$

$$\beta_2 = 2 \frac{\partial \overset{\mathbf{I}}{W}}{\partial \text{tr}(\boldsymbol{\tau})} = -\frac{\partial \overset{\mathbf{I}}{W}}{\partial I_{\tau_2}} - I_{\tau_1} \frac{\partial \overset{\mathbf{I}}{W}}{\partial I_{\tau_3}} + 2 \frac{\partial \overset{\mathbf{I}}{W}}{\partial J_3} + 2 \frac{\partial \overset{\mathbf{I}}{W}}{\partial J_4}, \quad (68)$$

$$\beta_3 = 3 \frac{\partial \overset{\mathbf{I}}{W}}{\partial \text{tr}(\boldsymbol{\tau}^3)} = \frac{\partial \overset{\mathbf{I}}{W}}{\partial I_{\tau_3}}. \quad (69)$$

Using equations (61) and (66) and the Cayley-Hamilton theorem (see Appendix B) we can rewrite the restriction (65) in the form

$$\text{tr}(\boldsymbol{\varepsilon} \boldsymbol{\tau}) = (\gamma_0 + \overset{\mathbf{I}}{W}) \text{tr} \boldsymbol{\varepsilon} + \gamma_1 \text{tr}(\boldsymbol{\varepsilon} \boldsymbol{\tau}) + \gamma_2 \text{tr}(\boldsymbol{\varepsilon} \boldsymbol{\tau}^2) \quad \text{for every } \boldsymbol{\tau} \text{ and } \boldsymbol{\varepsilon}, \quad (70)$$

where γ_0 , γ_1 and γ_2 are functions of $\alpha_1, \dots, \alpha_9$, β_1 , β_2 , β_3 , I_{τ_1} , I_{τ_2} and I_{τ_3} . Since equation (70) has to hold for every $\boldsymbol{\tau}$ and $\boldsymbol{\varepsilon}$ (for more details see the supplementary material of [11]), we obtain the three equations

$$\gamma_0 = -\overset{\mathbf{I}}{W}, \quad \gamma_1 = 1 \quad \text{and} \quad \gamma_2 = 0, \quad (71)$$

which can be written in matrix form as:

$$\mathbf{M} \cdot \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix} = \begin{pmatrix} -\overset{\mathbf{I}}{W} \\ 1 \\ 0 \end{pmatrix}, \quad (72)$$

where the matrix \mathbf{M} depends only on $\alpha_1, \dots, \alpha_9$, I_{τ_1} , I_{τ_2} and I_{τ_3} (the entries of \mathbf{M} are given explicitly in Appendix C). Since β_1 , β_2 and β_3 depend on $\overset{\mathbf{I}}{W}$, the above gives three linear partial differential equations for the single variable $\overset{\mathbf{I}}{W}$. This implies that if $\alpha_1, \dots, \alpha_9$ are unrestricted, $\overset{\mathbf{I}}{W}$ is over-prescribed. Hence, the only way to satisfy equation (72) is to restrict $\alpha_1, \dots, \alpha_9$, as we show in the following section.

4.3 The case of small initial stress

In this section, we assume that the initial stress $\boldsymbol{\tau}$ is small. Our approach is to take the equations (71) and expand them in powers of $\boldsymbol{\tau}$, neglecting $\mathcal{O}(\|\boldsymbol{\tau}\|^3)$ terms, where $\|\cdot\|$ can be the Frobenius norm or any other equivalent norm. With reference to equation (70), we note that γ_1 multiplies an $\mathcal{O}(\|\boldsymbol{\tau}\|)$ term and γ_2 multiplies an $\mathcal{O}(\|\boldsymbol{\tau}\|^2)$ term. Therefore, it is only necessary to expand γ_1 up to $\mathcal{O}(\|\boldsymbol{\tau}\|)$ and γ_2 up to $\mathcal{O}(\|\boldsymbol{\tau}\|^0)$. Upon doing so, we obtain

$$\beta_1 (\alpha_1 + 3\alpha_2 + \alpha_3 \text{tr} \boldsymbol{\tau}) + \beta_2 (\alpha_2 \text{tr} \boldsymbol{\tau} + \alpha_3 \text{tr}(\boldsymbol{\tau}^2)) + \beta_3 \alpha_2 \text{tr}(\boldsymbol{\tau}^2) = -\overset{\mathbf{I}}{W}, \quad (73)$$

$$\beta_1 (3(\alpha_3 + 1) + \alpha_6 \operatorname{tr} \boldsymbol{\tau} + 2\alpha_8) + \beta_2 (\alpha_1 + (\alpha_3 + 1) \operatorname{tr} \boldsymbol{\tau}) = 1, \quad (74)$$

$$2\beta_1\alpha_9 + 2\beta_2\alpha_8 + \beta_3\alpha_1 = 0. \quad (75)$$

Next, we expand $\alpha_1, \dots, \alpha_9$ up to $\mathcal{O}(\|\boldsymbol{\tau}\|^2)$:

$$\alpha_i = \alpha_{i0} + \alpha_{i1} \operatorname{tr} \boldsymbol{\tau} + \alpha_{i2} (\operatorname{tr} \boldsymbol{\tau})^2 + \alpha_{i3} \operatorname{tr}(\boldsymbol{\tau}^2) \quad \text{for } i = 1, 2, \dots, 9, \quad (76)$$

where the α_{ij} , $i = 1, \dots, 9$, $j = 0, \dots, 3$ are constants. For more details on linearising in terms of isotropic invariants see [7]. We also expand $\overset{\mathbf{I}}{W}$ up to $\mathcal{O}(\|\boldsymbol{\tau}\|^3)$:

$$\begin{aligned} \overset{\mathbf{I}}{W} = & \psi_0 + \psi_1 \operatorname{tr} \boldsymbol{\tau} + \psi_2 (\operatorname{tr} \boldsymbol{\tau})^2 + \psi_3 \operatorname{tr}(\boldsymbol{\tau}^2) + \psi_4 (\operatorname{tr} \boldsymbol{\tau})^3 \\ & + \psi_5 \operatorname{tr} \boldsymbol{\tau} \operatorname{tr}(\boldsymbol{\tau}^2) + \psi_6 \operatorname{tr}(\boldsymbol{\tau}^3), \end{aligned} \quad (77)$$

where ψ_0, \dots, ψ_6 are constants and we immediately choose $\psi_0 = 0$ since we expect

$$\lim_{\boldsymbol{\tau} \rightarrow \mathbf{0}} \overset{\mathbf{I}}{W} = 0. \quad (78)$$

Upon substituting equation (77) into equations (67)–(75), we obtain β_1 , β_2 and β_3 expanded up to $\mathcal{O}(\|\boldsymbol{\tau}\|^2)$, $\mathcal{O}(\|\boldsymbol{\tau}\|^1)$ and $\mathcal{O}(\|\boldsymbol{\tau}\|^0)$, respectively, which can then be substituted into equations (73)–(75). We then solve the resulting system of equations for the parameters α_{ij} and ψ_i , where we note that the stress tensor of an initially stressed material must generalise that derived from classical linear elasticity. In other words, when $\boldsymbol{\tau} \rightarrow \mathbf{0}$ we must have

$$\delta \boldsymbol{\sigma} = \alpha_{10} \boldsymbol{\varepsilon} + \alpha_{20} \mathbf{I} \operatorname{tr}(\boldsymbol{\varepsilon}), \quad \text{where } \alpha_{10} = 2\mu \quad \text{and} \quad \alpha_{20} = \lambda, \quad (79)$$

where λ and μ are the first and second Lamé parameters, respectively. Using equation (79), the final system of equations simplifies to the following conditions:

$$\psi_1 = 0, \quad \psi_2 = -\frac{\lambda}{12\kappa\mu}, \quad \psi_3 = \frac{1}{4\mu}, \quad (80)$$

$$\psi_4 = \frac{2\lambda^2(3\alpha_{11} - 2\alpha_{80}) + 2\lambda\mu(4\alpha_{11} + 4\alpha_{30} + 3) - 8\mu^2\alpha_{21}}{216\kappa^2\mu^2}, \quad (81)$$

$$\psi_5 = \frac{\lambda(2\alpha_{80} - 3\alpha_{11}) - 2\mu(\alpha_{11} + \alpha_{30} + 1)}{24\kappa\mu^2}, \quad \psi_6 = -\frac{\alpha_{80}}{6\mu^2}, \quad (82)$$

$$\alpha_{80} = \frac{2\mu\alpha_{30} - 3\kappa\alpha_{11}}{2\lambda}, \quad (83)$$

where $\kappa = \lambda + 2\mu/3$ is the bulk modulus of the material under consideration. Equation (83) relates α_{80} to λ , μ , α_{11} and α_{30} , and therefore reduces the

number of free parameters in the system by one. We now use the above to write the linearised Cauchy stress in terms of the strain and initial stress:

$$\boxed{\begin{aligned} \delta \boldsymbol{\sigma} = & \boldsymbol{\tau} + \boldsymbol{\omega} \boldsymbol{\tau} - \boldsymbol{\tau} \boldsymbol{\omega} + \mathbf{I} \operatorname{tr}(\boldsymbol{\varepsilon} \boldsymbol{\tau}) + 2(\mu + \mu_1 \operatorname{tr} \boldsymbol{\tau}) \boldsymbol{\varepsilon} + (\lambda + \lambda_1 \operatorname{tr} \boldsymbol{\tau}) \mathbf{I} \operatorname{tr}(\boldsymbol{\varepsilon}) \\ & + \eta (\boldsymbol{\tau} \operatorname{tr}(\boldsymbol{\varepsilon}) + \mathbf{I} \operatorname{tr}(\boldsymbol{\varepsilon} \boldsymbol{\tau})) + \left(\frac{\mu \eta}{\lambda} - \frac{3\kappa \mu_1}{2\lambda} \right) (\boldsymbol{\varepsilon} \boldsymbol{\tau} + \boldsymbol{\tau} \boldsymbol{\varepsilon}), \end{aligned}} \quad (84)$$

where we have renamed $\alpha_{11} = 2\mu_1$, $\alpha_{21} = \lambda_1$ and $\alpha_{30} = \eta$ and all the parameters in the equation above are constants. Equation (84) differs from the stress tensor first deduced in [27] because of the restriction given in equation (83). The parameters above may be further restricted by considerations such as strong-ellipticity [59, 14], but ultimately, they can be determined by ultrasonic, indentation, or hole drilling experiments.

4.3.1 Initially stressed neo-Hookean models

If equations (34) and (35) are expanded for small $\boldsymbol{\tau}$, they can be solved for K_1 and K_2 , which have the same series expansion up to order one in $\boldsymbol{\tau}$:

$$K_1 = K_2 = 1 + \frac{I_{\tau_1}}{3\kappa} + \mathcal{O}(\boldsymbol{\tau}^2). \quad (85)$$

Equation (85) can then be substituted into equations (62) and (63) to obtain

$$\alpha_1 = 2\mu - \frac{2(\lambda + \mu)}{3\kappa} I_{\tau_1} + \mathcal{O}(\boldsymbol{\tau}^2), \quad \alpha_2 = \lambda - \frac{\lambda}{3\kappa} I_{\tau_1} + \mathcal{O}(\boldsymbol{\tau}^2), \quad (86)$$

for the first model, and

$$\alpha_1 = 2\mu - \frac{2(\lambda + \mu)}{3\kappa} I_{\tau_1} + \mathcal{O}(\boldsymbol{\tau}^2), \quad \alpha_2 = \lambda + \frac{2\lambda}{3\kappa} I_{\tau_1} + \mathcal{O}(\boldsymbol{\tau}^2). \quad (87)$$

for the second. Therefore, for both models, we have

$$\alpha_{10} = 2\mu, \quad \alpha_{11} = -\frac{2(\lambda + \mu)}{3\kappa}, \quad \alpha_{20} = \lambda, \quad \alpha_{30} = -1, \quad \text{and} \quad \alpha_{80} = 1, \quad (88)$$

which satisfy equation (83), as required. The linearised stress tensors associated with the two models are

$$\delta \boldsymbol{\sigma}_{\text{GSC1}} = \boldsymbol{\tau} + \boldsymbol{\omega} \boldsymbol{\tau} - \boldsymbol{\tau} \boldsymbol{\omega} - \boldsymbol{\tau} \operatorname{tr}(\boldsymbol{\varepsilon}) + 2 \left(\mu - \frac{\lambda + \mu}{3\kappa} \operatorname{tr} \boldsymbol{\tau} \right) \boldsymbol{\varepsilon} \quad (89)$$

$$+ \left(\lambda - \frac{\lambda}{3\kappa} \operatorname{tr} \boldsymbol{\tau} \right) \mathbf{I} \operatorname{tr}(\boldsymbol{\varepsilon}) + \boldsymbol{\varepsilon} \boldsymbol{\tau} + \boldsymbol{\tau} \boldsymbol{\varepsilon}, \quad (90)$$

and

$$\delta\boldsymbol{\sigma}_{\text{GSC2}} = \boldsymbol{\tau} + \boldsymbol{\omega}\boldsymbol{\tau} - \boldsymbol{\tau}\boldsymbol{\omega} - \boldsymbol{\tau} \operatorname{tr}(\boldsymbol{\varepsilon}) + 2 \left(\mu - \frac{\lambda + \mu}{3\kappa} \operatorname{tr} \boldsymbol{\tau} \right) \boldsymbol{\varepsilon} \quad (91)$$

$$+ \left(\lambda + \frac{2\lambda}{3\kappa} \operatorname{tr} \boldsymbol{\tau} \right) \mathbf{I} \operatorname{tr}(\boldsymbol{\varepsilon}) + \boldsymbol{\varepsilon}\boldsymbol{\tau} + \boldsymbol{\tau}\boldsymbol{\varepsilon}. \quad (92)$$

5 Discussion

Most constitutive choices in the literature of the form $W := W(\mathbf{F}, \boldsymbol{\tau})$ do not satisfy the ISRI restrictions (12) and (65) presented in this paper. In Section 2.1 we gave an example of how these constitutive choices may lead to unphysical behaviour even for simple deformations such as uniaxial extension. This is also true of more complex deformations. Taking an example from biomechanics, where residual stresses play a crucial role, suppose we wish to model the mechanics of an arterial wall that supports an internal pressure. Let us choose two different reference configurations: first, the *unloaded configuration* where the fluid in the artery has been removed, and second, the *opening angle configuration* [41, 21] where the fluid has been removed and the artery has been cut along its axis. Both these configurations are subject to no external loads, but there will be less (and differently distributed) internal stress in the opening angle configuration. If we use a strain energy function $W(\mathbf{F}, \boldsymbol{\tau})$ that does not satisfy ISRI, then each of the two reference configurations will lead to a different stress distribution in the intact, inflated configuration of the arterial wall. We therefore cannot believe the predictions from either reference configuration since a physically correct model should *not* give different results due to an arbitrary choice of reference configuration.

By using ISRI we were able to derive a restricted form for the linear elastic stress tensor (84) in the case of small initial stress. This reduced form may ultimately improve material characterisation based on ultrasonic and indentation experiments. Many studies (see [27] and the references therein) have confirmed that a linearised stress tensor of the form given in equation (84) is well suited to fitting experimental data.

One outstanding problem for metals [58], biological soft tissues and other materials [29] is the difficulty in differentiating between the effects of structural anisotropy [53] and anisotropy caused by initial stress. The linear form of ISRI given in equation (65) will help to differentiate between these effects, as it dictates a specific dependency of the elastic stress on the initial stress. Nevertheless, future work should focus on developing the consequences of ISRI for materials with structural anisotropy. This will be particularly

important for collagenous soft tissues, which are known to be structurally anisotropic due to the presence of collagen fibres [49, 48]. Initial stresses in soft tissues can be significant [41, 22, 6], so assuming a small initial stress may not give accurate predictions. Currently, the internal stress in soft tissues is often measured by excising a sample and then estimating its initial deformation from a theoretically stress-free configuration. To measure stress in-vivo, non-invasive techniques need to be improved. Ultrasound techniques are among the most suitable and promising methods for measuring initial stress [4, 15], and the ISRI restrictions could ultimately improve them.

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A Deduction of the strain energy function W_{GCD}

The strain energy function (22) was first derived in [11]. Here, an alternative derivation is presented by considering deformations of an incompressible neo-Hookean material from a stress-free configuration \mathcal{B}_0 to the stressed configurations \mathcal{B} and $\bar{\mathcal{B}}$ (see Figure 3 and compare with Figure 2).

The neo-Hookean strain energy function is given by

$$W_{\text{NH}} = \mu(I_1 - 3), \quad (93)$$

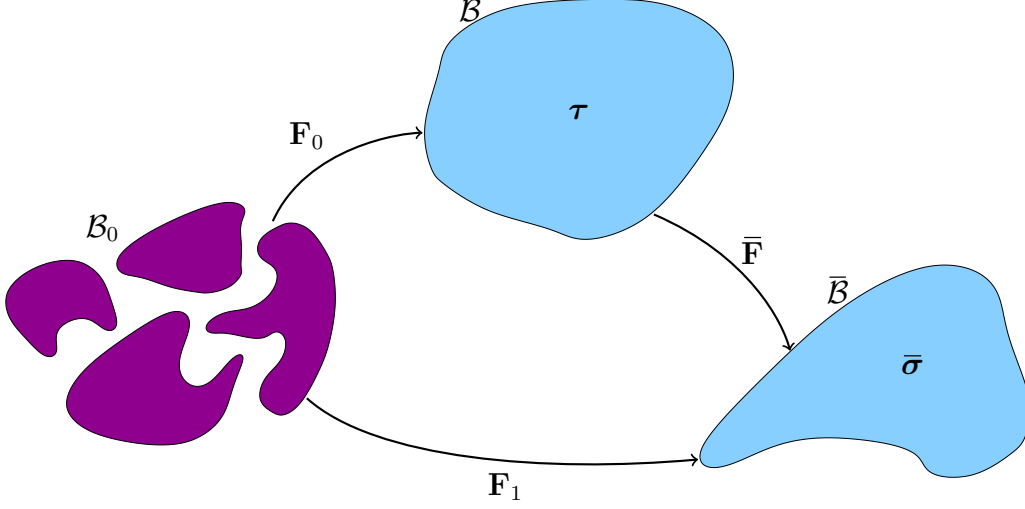


Figure 3: Deformation of an incompressible neo-Hookean material from a stress-free configuration \mathcal{B}_0 to the stressed configurations \mathcal{B} and $\bar{\mathcal{B}}$.

where μ is the ground state shear modulus of the material under consideration. Upon substituting (93) into (18) with $W_{I_3} = 0$ (because the material is incompressible) and then taking $\mathbf{F} = \mathbf{F}_0$ and $\mathbf{F} = \mathbf{F}_1$, it follows that

$$\boldsymbol{\tau} = \mu \mathbf{B}_0 - p_0 \mathbf{I} \quad \text{and} \quad \bar{\boldsymbol{\sigma}} = \mu \mathbf{B}_1 - p_1 \mathbf{I}, \quad (94)$$

where $\mathbf{B}_0 = \mathbf{F}_0 \mathbf{F}_0^T$, $\mathbf{B}_1 = \mathbf{F}_1 \mathbf{F}_1^T$ and p_0 and p_1 are the Lagrange multipliers associated with the two respective deformations. By rearranging equation (94)₁ and taking the determinant of both sides, the following is obtained:

$$\det(\mu \mathbf{B}_0) = \det(\boldsymbol{\tau} + p_0 \mathbf{I}) \quad \Leftrightarrow \quad \mu^3 = p_0^3 + p_0^2 I_{\tau_1} + p_0 I_{\tau_2} + I_{\tau_3}, \quad (95)$$

where $\det(\mathbf{B}_0) = 1$ because the material is incompressible. Only one of the three roots of the above polynomial is physically meaningful [11] and it is given by equation (23). Using $\mathbf{F}_1 = \bar{\mathbf{F}} \mathbf{F}_0$, equation (94)₂ gives

$$\bar{\boldsymbol{\sigma}} = \mu \bar{\mathbf{F}} \mathbf{B}_0 \bar{\mathbf{F}}^T - p_1 \mathbf{I}. \quad (96)$$

The aim is to derive an initially stressed strain energy function that gives equation (96) with \mathcal{B} as the reference configuration. For simplicity, it is assumed that the strain energy function depends only upon I_1 , J_1 and the three initial stress invariants I_{τ_1} , I_{τ_2} and I_{τ_3} . Making this assumption and substituting $\mathbf{F} = \bar{\mathbf{F}}$ into equation (18) with $W_{I_3} = 0$, it follows that

$$\bar{\boldsymbol{\sigma}} = \boldsymbol{\sigma}(\bar{\mathbf{F}}, \boldsymbol{\tau}) = 2W_1 \bar{\mathbf{B}} + 2W_{J_1} \bar{\mathbf{F}} \boldsymbol{\tau} \bar{\mathbf{F}}^T - \bar{p} \mathbf{I} \quad (97)$$

$$= 2W_1 \bar{\mathbf{B}} + 2W_{J_1} (\mu \bar{\mathbf{F}} \mathbf{B}_0 \bar{\mathbf{F}}^T - p_0 \bar{\mathbf{B}}) - \bar{p} \mathbf{I}. \quad (98)$$

For equation (98) to be equivalent to equation (96), the following equations must be satisfied:

$$2W_1 = p_0, \quad 2W_{J_1} = 1, \quad \bar{p} = p_1. \quad (99)$$

The third of these equations does not tell us anything about the required functional form of W ; however, upon solving the first two, the following is obtained:

$$W = \frac{1}{2}(p_0(I_{\tau_1}, I_{\tau_2}, I_{\tau_3})I_1 + J_1) + f(I_{\tau_1}, I_{\tau_2}, I_{\tau_3}), \quad (100)$$

where f is an arbitrary function of I_{τ_1} , I_{τ_2} and I_{τ_3} . Upon choosing $f(I_{\tau_1}, I_{\tau_2}, I_{\tau_3}) = -\frac{3}{2}\mu$, the final form of the strain energy function (22) is obtained. This choice ensures that the energy derived using the initially stressed strain energy function is the same as that obtained by considering a direct deformation of a neo-Hookean material from the stress-free configuration.

All that remains is to prove that, when using W_{GCD} , the third equation of (99) holds. Equations (94)₁ and (96) can be rearranged to give

$$p_0\mathbf{I} = \mu\mathbf{B}_0 - \boldsymbol{\tau} \quad \text{and} \quad p_1\mathbf{I} = \mu\bar{\mathbf{F}}\mathbf{B}_0\bar{\mathbf{F}}^T - \bar{\boldsymbol{\sigma}}, \quad (101)$$

respectively. Multiplying the first of these equations on the left by $\bar{\mathbf{F}}$ and on the right by $\bar{\mathbf{F}}^T$, and upon substituting equation (100) into equation (97) and equation (97) into equation (101)₂, we obtain

$$p_0\bar{\mathbf{B}} = \mu\bar{\mathbf{F}}\mathbf{B}_0\bar{\mathbf{F}}^T - \bar{\mathbf{F}}\boldsymbol{\tau}\bar{\mathbf{F}}^T \quad (102)$$

and

$$p_1\mathbf{I} = \mu\bar{\mathbf{F}}\mathbf{B}_0\bar{\mathbf{F}}^T - p_0\bar{\mathbf{B}} + \bar{p}\mathbf{I} - \bar{\mathbf{F}}\boldsymbol{\tau}\bar{\mathbf{F}}^T, \quad (103)$$

respectively. Then substituting (102) into (103), we obtain

$$p_1\mathbf{I} = \bar{p}\mathbf{I} \quad \Rightarrow \quad p_1 = \bar{p}, \quad (104)$$

as required.

B Tensor Identities

The Cayley-Hamilton theorem allows us to determine which tensors are independent. It states that any 3×3 tensor \mathbf{A} satisfies

$$\mathbf{A}^3 - I_{\mathbf{A}_1}\mathbf{A}^2 + I_{\mathbf{A}_2}\mathbf{A} - I_{\mathbf{A}_3}\mathbf{I} = \mathbf{0}, \quad (105)$$

where $I_{\mathbf{A}_1}$, $I_{\mathbf{A}_2}$ and $I_{\mathbf{A}_3}$ are the invariants of \mathbf{A} analagous to I_{τ_1} , I_{τ_2} and I_{τ_3} for $\boldsymbol{\tau}$. From equation (105), we can see that any power of $\boldsymbol{\tau}$ higher than two can be rewritten in terms of $\boldsymbol{\tau}^2$, $\boldsymbol{\tau}$, \mathbf{I} and the invariants I_{τ_1} , I_{τ_2} and I_{τ_3} .

We will now show that $\text{tr}(\boldsymbol{\tau}^2 \boldsymbol{\varepsilon})$ and $\boldsymbol{\tau} \boldsymbol{\varepsilon} \boldsymbol{\tau}$, $\boldsymbol{\tau}^2 \boldsymbol{\varepsilon} \boldsymbol{\tau} + \boldsymbol{\tau} \boldsymbol{\varepsilon} \boldsymbol{\tau}^2$ can be written as combinations of terms already present in equation (55). First substitute $\mathbf{A} = \boldsymbol{\varepsilon} + \gamma \boldsymbol{\tau}$ in equation (105), where γ is an arbitrary scalar. Since the resulting equation must hold for every γ , each coefficient multiplying a different power of γ must be zero individually. The term multiplying γ^2 is given by

$$\begin{aligned} \boldsymbol{\tau} \boldsymbol{\varepsilon} \boldsymbol{\tau} + \boldsymbol{\varepsilon} \boldsymbol{\tau}^2 + \boldsymbol{\tau}^2 \boldsymbol{\varepsilon} - (\boldsymbol{\varepsilon} \boldsymbol{\tau} + \boldsymbol{\tau} \boldsymbol{\varepsilon}) I_{\tau_1} - \boldsymbol{\tau}^2 \text{tr} \boldsymbol{\varepsilon} + \boldsymbol{\tau} (I_{\tau_1} \text{tr} \boldsymbol{\varepsilon} - \text{tr}(\boldsymbol{\varepsilon} \boldsymbol{\tau})) + \boldsymbol{\varepsilon} I_{\tau_2} \\ + \mathbf{I} (I_{\tau_1} \text{tr}(\boldsymbol{\tau} \boldsymbol{\varepsilon}) - I_{\tau_2} \text{tr} \boldsymbol{\varepsilon} - \text{tr}(\boldsymbol{\varepsilon} \boldsymbol{\tau}^2)) = 0. \end{aligned} \quad (106)$$

By taking the trace of both sides of this equation (and using the properties $\text{tr}(\mathbf{A} + \mathbf{B}) = \text{tr} \mathbf{A} + \text{tr} \mathbf{B}$ and $\text{tr}(\mathbf{AB}) = \text{tr}(\mathbf{BA})$) we establish that $\text{tr}(\boldsymbol{\tau}^2 \boldsymbol{\varepsilon})$ is indeed a combination of the terms already present in equation (55). The same can then be said for $\boldsymbol{\tau} \boldsymbol{\varepsilon} \boldsymbol{\tau}$ directly from equation (106), and for $\boldsymbol{\tau}^2 \boldsymbol{\varepsilon} \boldsymbol{\tau} + \boldsymbol{\tau} \boldsymbol{\varepsilon} \boldsymbol{\tau}^2$ by multiplying equation (106) on the left by $\boldsymbol{\tau}$.

C The entries of the matrix \mathbf{M}

The entries of the matrix \mathbf{M} are as follows:

$$M_{11} = \alpha_1 + 3\alpha_2 + \alpha_3 \text{tr} \boldsymbol{\tau}, \quad M_{12} = \alpha_2 \text{tr} \boldsymbol{\tau} + \alpha_3 \text{tr}(\boldsymbol{\tau}^2) + 2\alpha_9 I_{\tau_3}, \quad (107)$$

$$M_{13} = \alpha_2 \text{tr}(\boldsymbol{\tau}^2) + \alpha_3 \text{tr}(\boldsymbol{\tau}^3) + 2\alpha_8 I_{\tau_3} + 2\alpha_9 I_{\tau_1} I_{\tau_3}, \quad (108)$$

$$M_{21} = 3(\alpha_3 + 1) + \alpha_6 \text{tr} \boldsymbol{\tau} + 2\alpha_8, \quad (109)$$

$$M_{22} = \alpha_1 + (\alpha_3 + 1) \text{tr} \boldsymbol{\tau} + \alpha_6 \text{tr}(\boldsymbol{\tau}^2) - 2\alpha_9 I_{\tau_2}, \quad (110)$$

$$M_{23} = (\alpha_3 + 1) \text{tr}(\boldsymbol{\tau}^2) + \alpha_6 \text{tr}(\boldsymbol{\tau}^3) - 2\alpha_8 I_{\tau_2} + 2\alpha_9 (I_{\tau_3} - I_{\tau_1} I_{\tau_2}), \quad (111)$$

$$M_{31} = 2\alpha_9, \quad M_{32} = 2\alpha_8 + 2\alpha_9 \text{tr} \boldsymbol{\tau}, \quad (112)$$

$$M_{33} = \alpha_1 + 2\alpha_8 \text{tr} \boldsymbol{\tau} + 2\alpha_9 \text{tr}(\boldsymbol{\tau}^2). \quad (113)$$